Spin squeezing of mixed systems

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Abstract. The notion of spin squeezing has been discussed in this paper using the density matrix formalism. Extending the definition of squeezing for pure states given by Kitagawa and Ueda in an appropriate manner and employing the spherical tensor representation, we show that mixed spin states which are non-oriented and possess vector polarization indeed exhibit squeezing. We construct a mixed state of a spin 1 system using two spin 1/2 states and study its squeezing behaviour as a function of the individual polarizations of the two spinors.

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1. Introduction

The concept of spin is a fascinating topic in quantum theory. Defined through the commutation relations ($\hbar = 1$)

$$\vec{J} \times \vec{J} = i\vec{J}, \tag{1}$$

which are common to intrinsic spin $\vec{S}$ as well as orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$, it is interesting to note that equation (1), in the case of $\vec{L}$ can be derived from the position-momentum commutation relations

$$[k, p_k] = i \;, \quad k = x, y, z. \tag{2}$$

On the other hand, intrinsic spin associated with point particles like electrons are described in terms of the 'up' spinor, u and the 'down' spinor, v which are well-defined mathematically once the definition (1) for spin is accepted. Considering, therefore, the spinors as fundamental entities, Schwinger [2] visualized any state $|sm\rangle$ as made up of $s+m$ 'up' spinors and $s-m$ 'down' spinors through

$$|sm\rangle = \frac{(a_+^\dagger)^{s+m}(a_-^\dagger)^{s-m}}{[(s+m)!(s-m)!]^{\frac{1}{2}}}|00\rangle, \tag{3}$$

where $a_+^\dagger, a_-^\dagger$ are the creation operators for the spin 'up' and spin 'down' states, respectively. According to Biedenharn and Louck [3], this work of Schwinger was motivated by the celebrated paper of Majorana [4]. It is well-known that the fundamental uncertainty relation

$$\Delta x \, \Delta p_x \geq \frac{1}{2}, \tag{4}$$

of Heisenberg, which is equally valid for any pair of canonically conjugate variables, follows once (2) is postulated. Like wise, the uncertainty relations

$$\Delta S_\alpha^2 \, \Delta S_\beta^2 \geq \frac{1}{4}|\langle\psi|S_\gamma|\psi\rangle|^2, \tag{5}$$

for the spin operator $\vec{S}$, with $\alpha, \beta, \gamma = x, y, z$ cyclically, are derivable for any spin state $|\psi\rangle$ once (1) is postulated, although no two components of $\vec{S}$ are canonically conjugate to each other. It is also well-known that

$$\Delta x = \Delta p_x = \frac{1}{\sqrt{2}}, \tag{6}$$

in the case of the ground state of a simple harmonic oscillator in one dimension and this corresponds to the minimum uncertainty given by the equality in (4). A state for which

$$\Delta x < \frac{1}{\sqrt{2}}, \tag{7}$$

is then said to be squeezed in configuration space. One can similarly define a squeezed state of the oscillator in momentum space. Just as the idea of coherent states introduced by Schroedinger [5] for the harmonic oscillator was extended [6, 16] to discuss coherence in optics, the notion of squeezed states was also extended to the radiation field [7] and to spin states [1, 8, 9] as well. A spin state may be said [10] to be squeezed if the variance $\Delta S_\perp$ associated with a spin component normal to the mean spin direction $\hat{V}$ satisfies the condition,

$$\Delta S_\perp^2 < \frac{1}{2}\left|\langle\psi|\vec{S}\cdot\hat{V}|\psi\rangle\right|. \tag{8}$$

A more stringent condition

$$\xi = \left[\frac{2s(\Delta S_\perp)^2}{\left|\langle\psi|\vec{S}\cdot\hat{V}|\psi\rangle\right|^2}\right]^{\frac{1}{2}} < 1, \tag{9}$$

has been advocated by Wineland et al [9]. Kitagawa and Ueda [1] have argued that it would be possible to cancel out fluctuations in one direction normal to $\hat{V}$ at the expense of the other, provided quantum correlations are established among the elementary spinors which constitute a spin s state in the sense of (3). Likewise a physical basis for (9) has also been discussed by Puri [8]. More recently [10], a classification of pure states $|\psi\rangle$ into two classes referred to as 'oriented' and 'non-oriented' has been suggested employing a construction of states of spin s out of $2s$ non-collinear spinors and it was explicitly shown in the case of $s = 1$ that a state $|\psi\rangle$ has to be necessarily non-oriented for it to be a squeezed state.

The purpose of this paper is to extend the notion of spin squeezing to statistical assemblies of particles with spin s as it will not only provide a complete spin description of spin squeezing but also facilitate planning of experiments to study squeezed spin states. This discussion is best done naturally by employing the language of the density matrix. An advantage of the density matrix formalism is that it can be applied with equal ease to discuss pure as well as the mixed spin systems. This formalism is outlined in section 2 using the well known spherical tensor representation for the density matrix. In section 3 the squeezing condition (8) based on the uncertainty relation (5) is generalized to take care of statistical assemblies as well. In section 4, we show that squeezing is exhibited by only non-oriented systems with non-zero vector polarization. In section 5 we discuss the squeezing behaviour of a mixed spin 1 state which naturally arises in experiments [11] employing polarized spin $\frac{1}{2}$ beams on polarized spin $\frac{1}{2}$ targets. We also look into the spin-spin correlations which exist between these spinors when they are combined to yield a spin 1 state.

2. Density Matrix Description

The density matrix ρ for a spin s system, pure or mixed has the standard expansion

$$\rho = \frac{\mathrm{Tr}\rho}{2s+1}\sum_{k=0}^{2s}\sum_{q=-k}^{k}(-1)^q t^k_{-q}\,\tau^k_q, \tag{10}$$

where τ^k_q (with $\tau^0_0 = I$, the identity operator) are irreducible tensor operators of rank k in the $n = 2s+1$ dimensional spin space with projection q along the axis of quantization in the real 3-dimensional space. The τ^k_q satisfy the commutation relations

$$\left[\tau^{k_1}_{q_1}, \tau^{k_2}_{q_2}\right] = [s][k_1][k_2]\sum_{k=|k_1-k_2|}^{k_1+k_2}\left(1-(-1)^{k_1+k_2-k}\right)\,C(k_1k_2k; q_1q_2q) \\ \times W(sk_1sk_2; sk)\,\tau^k_q \tag{11}$$

where C and W denote Clebsch-Gordan and Racah coefficients respectively and we use the short hand $[s] = \sqrt{2s+1}$. They also satisfy the orthogonality relations

$$\mathrm{Tr}\{\tau^{k^\dagger}_q\,\tau^{k'}_{q'}\} = n\,\delta_{kk'}\delta_{qq'}. \tag{12}$$

Here the normalization has been chosen so as to be in agreement with Madison convention [12]. The Fano statistical tensors or the spherical tensor parameters t^k_q in (10) which characterize the given system are the average expectation values given by

$$t^k_q = \mathrm{Tr}\{\rho\tau^k_q\}/\mathrm{Tr}\rho. \tag{13}$$

Since ρ is Hermitian, and $\tau_q^{k^\dagger} = (-1)^q \tau_{-q}^k$, these satisfy the condition

$$t_q^{k^\star} = (-1)^q t_{-q}^k. \tag{14}$$

Apart from $t_0^0 = \mathrm{Tr}\rho$, there are $n^2 - 1 = 4s(s+1)$ real independent parameters for the most general mixed state. The density matrix ρ for pure state satisfies $\rho^2 = \rho$, and hence a normalized pure state has only $4s$ real independent parameters describing it. This leads to a set of constraints

$$[k] \sum_{k_1,k_2} [k_1][k_2]\, W(sk_1sk_2; sk)\, (t^{k_1} \otimes t^{k_2})_q^k = [s]\, t_q^k \tag{15}$$

on t_q^k for each k and q. It is worth noting here that in addition to the above representation for density matrix which uses spherical tensor operators, there also exist other representations such as the $SU(n)$ representation [13], where the density matrix is expanded in terms of the generators of the Lie group $SU(n)$,whose number is also $n^2 - 1$. This representation is advantageous since the diagonal form of ρ can be expressed in terms of the subset consisting of diagonal generators which are $n-1$ in number. On the other hand, the spherical tensor representation which is widely used in spin physics has the advantage that the spherical tensor parameters have simple transformation properties under coordinate rotations in the real 3-dimensional space. If a coordinate frame I is transformed to II through a rotation $R(\alpha, \beta, \gamma)$,where α, β, γ are the Eulerian angles, the t_q^k in the respective frames are related through

$$(t_q^k)_{II} = \sum_{q'} D_{q'q}^k(\alpha, \beta, \gamma)(t_{q'}^k)_I, \tag{16}$$

where $D_{q'q}^k(\alpha, \beta, \gamma)$ is the matrix representation of the rotation. The spherical tensor operators τ_q^k are homogeneous polynomials of rank k and projection q, constructed out of the spin operators S_x, S_y and S_z. In particular, the operator $\vec{S}$ is a vector (rank 1) operator and its spherical components are related to τ_q^1 through

$$S_q^1 = \left[\frac{s(s+1)}{3}\right]^{\frac{1}{2}} \tau_q^1\ ;\ q = 1, 0, -1. \tag{17}$$

The average expectation value of $\vec{S}$ in the state specified by ρ given by

$$\vec{P} = \frac{\mathrm{Tr}\{\rho\vec{S}\}}{\mathrm{Tr}\rho} \tag{18}$$

is called the vector polarization in spin physics literature. Kitagawa and Ueda [1] refer to this as the *mean spin vector* in their paper. The spherical components of $\vec{P}$,

$$P_{\pm 1} = \mp\frac{1}{\sqrt{2}}(P_x \pm iP_y)\ ;\ P_0 = P_z, \tag{19}$$

are related to t_q^1 through

$$P_q^1 = \frac{\mathrm{Tr}\{\rho S_q^1\}}{\mathrm{Tr}\rho} = \left[\frac{s(s+1)}{3}\right]^{\frac{1}{2}} t_q^1. \tag{20}$$

The expectation values of other observables such as S_x^2, S_y^2, S_z^2 on the other hand are related to the alignment parameters t_q^2 through

$$\mathrm{Tr}\{\rho S_x^2\} = \tfrac{1}{f_1^2} - \tfrac{1}{f_2\sqrt{6}}\, t_0^2 + \tfrac{1}{2f_2}(t_2^2 + t_{-2}^2) \tag{21}$$

$$\mathrm{Tr}\{\rho S_y^2\} = \tfrac{1}{f_1^2} - \tfrac{1}{f_2\sqrt{6}}\, t_0^2 - \tfrac{1}{2f_2}(t_2^2 + t_{-2}^2) \tag{22}$$

$$\mathrm{Tr}\{\rho S_z^2\} = \tfrac{1}{f_1^2} + \tfrac{1}{f_2}\sqrt{\tfrac{2}{3}}\, t_0^2, \tag{23}$$

where

$$f_1 = \left[\frac{3}{s(s+1)}\right]^{\frac{1}{2}} \; ; \; f_2 = \left[\frac{30}{s(s+1)(2s-1)(2s+3)}\right]^{\frac{1}{2}}. \tag{24}$$

These lead to the variances

$$\begin{aligned}\Delta S_x^2 &= \frac{\mathrm{Tr}\{\rho S_x^2\}}{\mathrm{Tr}\rho} - \left[\frac{\mathrm{Tr}\{\rho S_x\}}{\mathrm{Tr}\rho}\right]^2 \\ &= \frac{1}{\mathrm{Tr}\rho}\left[\tfrac{1}{f_1^2} - \tfrac{1}{\sqrt{6}f_2}t_0^2 + \tfrac{1}{2f_2}(t_2^2 + t_{-2}^2)\right] - \tfrac{1}{2f_1^2}\frac{1}{(\mathrm{Tr}\rho)^2}(t_{-1}^1 - t_1^1)^2 \end{aligned} \tag{25}$$

$$\begin{aligned}\Delta S_y^2 &= \frac{\mathrm{Tr}\{\rho S_y^2\}}{\mathrm{Tr}\rho} - \left[\frac{\mathrm{Tr}\{\rho S_y\}}{\mathrm{Tr}\rho}\right]^2 \\ &= \frac{1}{\mathrm{Tr}\rho}\left[\tfrac{1}{f_1^2} - \tfrac{1}{\sqrt{6}f_2}t_0^2 - \tfrac{1}{2f_2}(t_2^2 + t_{-2}^2)\right] + \tfrac{1}{2f_1^2}\frac{1}{(\mathrm{Tr}\rho)^2}(t_{-1}^1 + t_1^1)^2 \end{aligned} \tag{26}$$

$$\begin{aligned}\Delta S_z^2 &= \frac{\mathrm{Tr}\{\rho S_z^2\}}{\mathrm{Tr}\rho} - \left[\frac{\mathrm{Tr}\{\rho S_z\}}{\mathrm{Tr}\rho}\right]^2 \\ &= \frac{1}{\mathrm{Tr}\rho}\left[\tfrac{1}{f_1^2} - \tfrac{1}{f_2}\sqrt{\tfrac{2}{3}}t_0^2\right] - \tfrac{1}{f_1^2}\frac{1}{(\mathrm{Tr}\rho)^2}(t_0^1)^2. \end{aligned} \tag{27}$$

While a system in a pure state satisfying $\rho^2 = \rho$ is completely polarized, a system in a mixed state is either partially polarized or unpolarized. For an unpolarized system, $t_q^k = 0$ for all $k = 1, \ldots, 2s$. A partially polarized or completely polarized state is said to be vector polarized if $\vec{P} \neq 0$ and aligned or tensor polarized if at least one $t_q^2 \neq 0$. A Cartesian coordinate frame chosen with its $\hat{z}-$axis parallel to $\vec{P}$ is referred to as Lakin Frame [LF] [14]. In other words, in such a frame $t_{\pm 1}^1 = 0$. On the other hand, a frame in which t_2^2 is real and $t_{\pm 1}^2 = 0$ is referred to as the Principal Axes of Alignment Frame (PAAF) [15]. The latter is also identified as a frame in which the traceless second rank Cartesian tensor $P_{\alpha\beta}$ (which is defined by the t_q^k) is diagonal. While there is only one PAAF for a system up to possible renaming of the axes, Lakin frame on the other hand requires only $\hat{z}_0$ axis to be along $\hat{P}$ and depending on the choice of x_0 and y_0 axes, we have an infinite number of LFs.

3. Spin squeezing

The Heisenberg uncertainty relationship for the spin operators S_x, S_y and S_z satisfying (1) is given by (5), where the variance ΔS_i^2 and the average expectation value $\langle S_z \rangle$ depend not only on the spin state but also on the frame with respect to which the spin operators have been specified. Following Kitagawa and Ueda [1] and Puri [8], we have defined squeezing criterion in our earlier paper [10] as follows. Given the quantum state $|\psi\rangle$ of spin s, the mean spin direction associated with it is given by

$$\hat{P} = \frac{\langle\psi|\vec{S}|\psi\rangle}{\left|\langle\psi|\vec{S}|\psi\rangle\right|}. \tag{28}$$

The state $|\psi\rangle$ is then said to be squeezed in the spin component $S_\perp = \vec{S}\cdot\hat{P}_\perp$, if

$$(\Delta S_\perp)^2 < \frac{1}{2}\left|\langle\vec{S}\cdot\hat{P}\rangle\right|, \tag{29}$$

where $\hat{P}_{\perp}$ is orthogonal to $\hat{P}$. This criterion of squeezing aims at characterizing squeezing as an intrinsic feature of the state and Kitagawa and Ueda [1] have remarked that if the spin state is visualized as being made up of $2s$ spin $\frac{1}{2}$ states, then the quantum correlations that exist among these component spins are responsible for the manifestation of squeezing in the given quantum state. They substantiate their statement through a pictorial representation in which they show that a spin coherent state which is built out of $2s$ spinors all oriented in the same direction, is not squeezed as there exist no quantum correlations in such an arrangement. On the other hand, a squeezed state of spin is depicted as being built out of the same number of the spins which possess quantum correlation. We have looked into this aspect, in all its details, in the case of spin 1 in our earlier paper [10] and an explicit connection between the spin-spin correlations and spin squeezing has been shown to exist. In the light of this we now adopt for the case of mixed states the generalized form of the above criterion. Explicitly, a spin state specified by ρ is said to be squeezed in the component $S_{\perp}\,(\equiv \vec{S}\cdot\hat{P}_{\perp})$, if

$$\Delta(\vec{S}\cdot\hat{P}_{\perp})^2 = \frac{\mathrm{Tr}\{\rho(\vec{S}\cdot\hat{P}_{\perp})^2\}}{\mathrm{Tr}\rho} < \frac{1}{2}\left|\langle\vec{S}\cdot\hat{P}\rangle\right| = \frac{|\mathrm{Tr}\{\rho\vec{S}\cdot\hat{P}\}|}{2\mathrm{Tr}\rho} \tag{30}$$

where $\hat{P}_{\perp}$ denotes any direction which is orthogonal to the vector polarization $\vec{P}$. It may be noted here that the squeezing criterion defined here is distinct from several others used in the literature [16]. For example, if one uses the criterion

$$\Delta S_i^2 < \frac{1}{2}|\langle S_z\rangle|\ ;\ i = x, y, \tag{31}$$

where the components are referred to a frame chosen arbitrarily then, as has been pointed out by Kitagawa and Ueda [1], it turns out that a given state will be squeezed with respect to a component in one frame but will not be so in another frame. This aspect makes squeezing solely frame-dependent and extrinsic to the system. On the other hand, the above form (30) of the criterion suggests that a quantum state itself specifies a direction with respect to which it reveals the presence of squeezing in its spin component. As such, this criterion, which we adopt here, characterizes any manifestation of squeezing as an intrinsic property of a spin system, like in the case of radiation field. We now classify, as in our earlier paper, the spin states into oriented and non-oriented states and study the squeezing aspect of states in each class based on this criterion (30).

4. Mixed state classification and squeezing

4.1. Oriented system

A spin system is said to be oriented [17] if its density matrix ρ has a diagonal form $\rho_{_0}$ with its eigen states being the angular momentum basis states $|sm\rangle_{_0}$ referred to the axis of quantization $\hat{z}_{_0}$. In other words, an oriented system is one in which the populations are distributed with respect to the basis states $|sm\rangle_{_0}$ and $\hat{z}_{_0}$ is then called the axis of orientation. It may be noted here that this definition for mixed states is a natural generalization of the definition of an oriented pure state defined in our earlier paper [10]. If p_m denote the fractional populations of an oriented system in the states $|sm\rangle_{_0}$, the density matrix $\rho_{_0}$ is given by the expansion

$$\rho_{_0} = \sum p_m |sm\rangle_{_0}\,{}_{_0}\langle sm|,\ p_m \geq 0\ ;\ \sum p_m = 1. \tag{32}$$

The vector polarization $\vec{P}$, for such a system turns out to be along $\hat{z}_0$ itself and in a LF whose $\hat{z}-$axis is along $\hat{z}_0$, we have

$$\vec{P} = \left(0, 0, \sum_m m\, p_m\right) = \left(\sum_m m\, p_m\right) \hat{z}_0. \tag{33}$$

Any vector $\hat{P}_\perp$ perpendicular to $\hat{P}$, therefore lies in the xy-plane of the chosen LF and we can express it as

$$\hat{P}_\perp = \hat{x}\cos\phi + \hat{y}\sin\phi\ ;\ 0 \leq \phi < 2\pi. \tag{34}$$

This makes

$$\vec{S}\cdot\hat{P}_\perp = S_x\cos\phi + S_y\sin\phi \tag{35}$$

and we have

$$\langle\vec{S}\cdot\hat{P}_\perp\rangle = \langle S_x\rangle\cos\phi + \langle S_y\rangle\sin\phi = 0 \tag{36}$$

since $\langle S_x\rangle = \langle S_y\rangle = 0$ in the Lakin frame. The variance in $\vec{S}\cdot\hat{P}_\perp$ then becomes

$$\Delta(\vec{S}\cdot\hat{P}_\perp)^2 = \frac{1}{2}\left[s(s+1) - \sum_m m^2 p_m\right], \tag{37}$$

so that the squeezing criterion (30) for an oriented system takes the form

$$s(s+1) - \sum_m m^2\, p_m < \left|\sum_m m\, p_m\right|, \tag{38}$$

and it is quite easy to see that this inequality is never satisfied for any value of s. Thus we arrive at the conclusion that no oriented system, either pure or mixed, is squeezed . It is interesting to note here that every spin $\frac{1}{2}$ state, either pure or mixed, is always oriented. This is due to the property that any spin $\frac{1}{2}$ density matrix can always be diagonalized by an appropriate unitary matrix belonging to the group $SU(2)$, the latter bearing the property that it provides a representation of the rotation group in 3-dimension. In other words the eigen states of the density matrix ρ for a spin $\frac{1}{2}$ system can always be identified as the spin -up and spin down states with respect to an appropriate axis of quantization. This, together with what has been said above, implies that squeezing is absent in the case of spin $\frac{1}{2}$, irrespective of whether the state is pure or mixed. We now look at the second class of spin states namely the *non-oriented* spin systems and look at their squeezing behaviour in what follows.

4.2. Non-oriented system (states)

A non-oriented spin s state $|\psi\rangle$ has been defined earlier [13] as one which can not be identified as eigen state of S_z with any choice of $\hat{z}-$axis as the axis of quantization. We may therefore define a mixed non-oriented system as one where the eigen states $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle$ of the density matrix ρ can not all be identified with states $|sm\rangle,\ m = -s, \ldots, s$ with respect to any suitable $\hat{z}-$axis. In other words, at least one of the eigen states $|\psi_i\rangle,\ i = 1, \ldots, n$ has to be non-oriented as defined earlier. The system will be maximally non-oriented if every one of the eigen states is non-oriented. Such non-oriented systems can exist only for spin $s \geq 1$ since the unitary group in n-dimension is homomorphic to the rotation group in 3-dimensions only in the particular case of $n = 2$.

While an oriented system gets specified through the distribution of populations in angular momentum states with respect to a single axis namely the axis of orientation, it has been shown by Ramachandran and Ravishankar [18] that a non-oriented system is characterised by more than one axis. This identification has been arrived at using the spherical tensor representation for the density matrix of such a system. In the most general case it has been shown by them that a set of $s(2s+1)$ axes $\hat{Q}_i, i = 1, \ldots, s(2s+1)$ are needed to characterize a non-oriented state. Each t^k_q in (10) for any given ρ can be expressed as

$$t^k_q = r_k \Big(\ldots (Y_1(\hat{Q}_1) \otimes Y_1(\hat{Q}_2))^2 \otimes \ldots)^{k-1} \otimes Y_1(\hat{Q}_k) \Big)^k_q, \qquad (39)$$

where r_k is a real constant and $Y_{1m}(\hat{Q}_i)$ is the spherical harmonic function associated with the direction $\hat{Q}_i$. An oriented system, in this language, is one for which all the $s(2s+1)$ axes merge together to give a single axis $\hat{Q}_0$, which is itself the axis of orientation. The choice of $\hat{z}-$axis along $\hat{Q}_0$ for an oriented system leads to the vanishing of all t^k_q with $q \neq 0$ and therefore, an oriented system in its LF is described by the Fano statistical tensors t^k_0 only. The non-oriented systems, on the other hand, possess, in general, non-zero t^k_q with respect to any angular momentum basis.

Before we look into the aspect of squeezing, it may be appropriate to briefly mention the nomenclature for the specific kinds of spin systems, often adopted in the spin physics literature. A spin system with non-zero t^1_q is said to be vector polarized while that with non-zero t^2_q is said to be aligned. A purely aligned system has non-zero t^2_q but all other tensor polarizations including the vector polarization t^1_q will be zero.

Coming back to the notion of squeezing, it is to be noted that for the squeezing criterion to be satisfied, the system should necessarily possess non-zero vector polarization since only then the right hand side of the inequality in (30) will be non-zero and one can look for the satisfiability of the squeezing criterion. If the vector polarization is zero then every frame qualifies to be a Lakin frame and since $\Delta(\vec{S}\cdot\hat{i})^2$ is always non-negative for any direction $\hat{i}$, we conclude that all non-oriented states which do not possess vector polarization lack squeezing. One can, however, define higher order squeezing behaviour via a proper criterion and examine such situations. Having ruled out squeezing in the case of oriented systems and in the case of non-oriented systems with zero vector polarization, we are left with non-oriented systems which possess non-zero vector polarization. Let us suppose that the density matrix of such a system is specified with respect to the angular momentum basis $|sm\rangle$ relative to a frame xyz in terms of the spherical tensors t^k_q through (10). We now make a transition to a particular LF $x_0y_0z_0$ in the following way. The vector polarization direction $\hat{P} = \hat{z}_0$ of the system is determined by using (18). If (θ_0, ϕ_0) denote the direction of $\hat{z}_0$ with respect to xyz, the frame xyz is rotated first about the $\hat{z}-$axis through ϕ_0 and then about the new $\hat{y}-$axis through θ_0. The frame so obtained (call it $x'y'z_0$) is a Lakin frame as the $\hat{z}-$axis of xyz now coincides with $\hat{z}_0$. The spherical tensor parameters t^k_q that specify the density matrix ρ in this frame are related to t'^k_q through

$$t^k_q = \sum D^k_{q'q}(\phi_0, \theta_0, 0)\, t'^k_q. \qquad (40)$$

While this frame is enough for the purpose of identifying squeezing, we wish to use the additional degree of freedom of rotating $x'y'z_0$ about $\hat{z}_0$ through an angle γ to get a special LF. The angle γ here is so chosen that the second rank tensor t^2_2 after the rotation is real. With this choice of $x_0y_0z_0$, $t^1_1 = t^1_{-1} = 0$ and $t^2_2 = t^2_{-2}$, the first being due to transition to a LF while the second being due to the use of additional degree of freedom of rotation about $\hat{z}_0$ through α. In the special LF $x_0y_0z_0$, we then collect the relevant quantities needed for identifying squeezing given by

$$\langle S_{z_0}\rangle = \frac{1}{f_1} t^1_0 \quad ; \quad \langle S_{x_0}\rangle = \langle S_{y_0}\rangle = 0 \qquad (41)$$

$$\Delta S_{x_0}^2 = \frac{1}{f_1^2} + \frac{1}{2f_2}\left(2t_2^2 - \sqrt{\tfrac{2}{3}}t_0^2\right) \tag{42}$$

$$\Delta S_{y_0}^2 = \frac{1}{f_1^2} - \frac{1}{2f_2}\left(2t_2^2 + \sqrt{\tfrac{2}{3}}t_0^2\right). \tag{43}$$

Defining $S_\perp$ as in equation (29) we get

$$\Delta S_\perp^2 = \Delta S_{x_0}^2 \cos^2\phi + \Delta S_{y_0}^2 \sin^2\phi \tag{44}$$

so that the squeezing criterion for $S_\perp$ takes the form

$$1 + \left[\frac{3(2s+3)(2s-1)}{s(s+1)40}\right]^{\frac{1}{2}} \left(2t_2^2\cos 2\phi - \sqrt{\tfrac{2}{3}}t_0^2\right) \;<\; \frac{1}{2}\left[\frac{3}{s(s+1)}\right]^{\frac{1}{2}} |t_0^1|, \tag{45}$$

for any value of $\phi,\ 0 \le \phi \le 2\pi$. States satisfying this criterion are then squeezed for those $\phi,\ 0 \le \phi \le 2\pi$ in the component of spin. In specific cases, this inequality is indeed satisfied over a range of values for t_q^k and we therefore conclude that squeezing is indeed exhibited by only non-orenited states with non-zero vector polarization. To support the claim made here we present below in table (1) several situations which reveal the presence of squeezing.

We wish to note here that t_q^k present in the table above actually correspond to realistic situations as they have been chosen in accordance with the positive semi-definiteness requirement of the density matrix ρ. For example, for $s = 1$, this property of ρ implies that the spherical tensor parameters have to satisfy the boundary conditions [19]

$$0 \le \frac{1}{3}(1 \pm \sqrt{\tfrac{3}{2}}\, t_0^1 + \tfrac{1}{\sqrt{2}}\, t_0^2) \le 1 \tag{46}$$

$$0 \le \frac{1}{3}(1 - \sqrt{2}\, t_0^2) \le 1 \tag{47}$$

$$0 \le (t_0^1)^2 + 2|t_2^2|^2 + 2|t_1^2|^2 + (t_0^2)^2 \le 2 \tag{48}$$

$$0 \le \det\rho \le \frac{1}{27} \tag{49}$$

If all the t_q^ks are treated as the component of a $(2s+1)^2 - 1$ dimensional complex vector $\vec{T}$, then, when the spin system is subjected to an interaction, this vector $\vec{T}$ starts moving in the $(2s+1)^2 - 1$ dimensional complex space, of course, subjected to the above constraints. It is therefore natural to ask how the squeezing evolves during such an evolution. This behaviour merits an independent study which is being taken up at present.

5. Squeezing of channel spin 1 states

The concept of channel spin plays an important role in hadron scattering and reaction processes. Consider for example, a beam of nucleons colliding with a proton target both of which are prepared initially to be mixed states specified by their density matrices

$$\rho(i) = \frac{1}{2}\left[1 + \vec{\sigma}(i)\cdot\vec{P}(i)\right] = \frac{1}{2}\sum_{k,q} t_q^k(i)\tau_q^{k^\dagger}(i)\ ;\ i = 1, 2. \tag{50}$$

Channel spin states $s = 0, 1$ come into play in scattering and reaction process [20]. The combined density matrix ρ_c is the direct product of these two density matrices, i.e.,

$$\rho_c = \rho_1 \otimes \rho_2, \tag{51}$$

Figure 1. Special Lakin Frame $x_0 y_0 z_0$, where $\hat{z}_0$ is along $\vec{P}(1)+\vec{P}(2)$, $\hat{x}_0-$ axis in the plane of $\vec{P}(1), \vec{P}(2)$ such that the azimuths of $\vec{P}(1), \vec{P}(2)$ are $0, \pi$ respectively.

and the density matrix for the channel spin 1 state is given by

$$\rho = \left[\frac{3+\vec{P}(1)\cdot\vec{P}(2)}{12}\right]\left[1+\sum_{k,q} t^k_q \tau^{k^\dagger}_q\right], \tag{52}$$

where the spherical tensor parameters t^k_q are related to the individual $t^k_q(i)$ through

$$t^k_q = \mathrm{Tr}(\rho\, \tau^k_q) = \left[\frac{6\sqrt{3}}{3+\vec{P}(1)\cdot\vec{P}(2)}\right]\sum_{k_1,k_2}[k_1][k_2]\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & k_1 \\ \frac{1}{2} & \frac{1}{2} & k_2 \\ 1 & 1 & k \end{Bmatrix}\left(t^{k_1}(1)\otimes t^{k_2}(2)\right)^k_q. \tag{53}$$

In equation (53), $\{\}$ denotes the Wigner 9-j symbol [21]. Explicitly we have

$$t^1_q = \left[\frac{\sqrt{6}}{3+\vec{P}(1)\cdot\vec{P}(2))}\right](\vec{P}_q(1)+\vec{P}_q(2)) \tag{54}$$

$$t^2_q = \left[\frac{2\sqrt{3}}{3+\vec{P}(1)\cdot\vec{P}(2))}\right](\vec{P}(1)\otimes\vec{P}(2))^2_q. \tag{55}$$

In order to discuss the squeezing nature of the channel state, we have to first choose a Lakin frame. A glance at the above equation for t^1_q suggests that the $\hat{z}_0-$axis (of the LF) should be chosen along $\vec{P}(1)+\vec{P}(2)$. Since $\vec{P}(1), \vec{P}(2)$ together define a plane in any general situation, we choose $\hat{x}_0$ axis to be in this plane such that the azimuths of $\vec{P}(1), \vec{P}(2)$ with respect to $\hat{x}_0$ are respectively 0 and π. The y_0 axis is then chosen to be along $\hat{z}_0\times\hat{x}_0$. The frame so chosen is indeed the Special LF (see figure 1) as is evident from equations (54) and (55) that $t^1_{\pm1}=0$ and $t^2_2 = t^2_{-2}$. In this frame so chosen, we have

$$P_{x_0}(1) = \frac{P(1)P(2)\sin\theta}{|\vec{P}(1)+\vec{P}(2)|} = -P_{x_0}(2) \tag{56}$$

$$P_{y_0}(1) = P_{y_0}(2) = 0 \tag{57}$$

$$P_{z_0}(1) = \frac{P(1)^2+P(1)P(2)\cos\theta}{|\vec{P}(1)+\vec{P}(2)|}\ ;\ P_{z_0}(2) = \frac{P(2)^2+P(1)P(2)\cos\theta}{|\vec{P}(1)+\vec{P}(2)|}. \tag{58}$$

If now $S_\perp$ is defined as $S_{x_0}\cos\phi + S_{y_0}\sin\phi$, then the variance $\Delta S^2_\perp$ takes the form

$$\Delta S^2_\perp = \frac{2[|\vec{P}(1)+\vec{P}(2)|^2 - P(1)^2P(2)^2\sin^2\theta\cos^2\phi]}{(3+\vec{P}(1)\cdot\vec{P}(2))|\vec{P}(1)+\vec{P}(2)|^2}, \tag{59}$$

while the expectation value of S_{z_0} will be given by

$$\langle S_{z_0}\rangle = \frac{2|\vec{P}(1)+\vec{P}(2)|}{(3+\vec{P}(1)\cdot\vec{P}(2))}. \tag{60}$$

The squeezing condition for $S_\perp$ then becomes

$$1 - \frac{\left|\vec{P}(1)\times\vec{P}(2)\right|^2}{\left|\vec{P}(1)+\vec{P}(2)\right|^2}\ \cos^2\phi < \frac{1}{2}|\vec{P}(1)+\vec{P}(2)|. \tag{61}$$

Figure 2. Variation of squeezing in S_{x_0} with respect to θ, the angle between the two polarization vectors $\vec{P}(1)$ and $\vec{P}(2)$.

Figure 3. Variation of squeezing in $S_{\perp}$ with respect to θ, the angle between the two polarization vectors $\vec{P}(1)$ and $\vec{P}(2)$.

This expression has been studied numerically for several cases of $\vec{P}(1), \vec{P}(2)$ and ϕ. Squeezing is seen for a wide range of values of $\vec{P}(1)$, $\vec{P}(2)$ and ϕ and in particular, maximum squeezing occurs when $\phi = 0$ for any given $\vec{P}(1), \vec{P}(2)$. In other words, it is the spin component S_{x_0} of the Special LF which is maximally squeezed. A plot of the quantity

$$Q = \frac{1}{2}|\langle S_{z_0}\rangle| - \Delta S_{x_0}^2 = \frac{1}{2}|\vec{P}(1)+\vec{P}(2)| + \frac{\left|\vec{P}(1)\times\vec{P}(2)\right|^2}{\left|\vec{P}(1)+\vec{P}(2)\right|^2}\,\cos^2\phi - 1 \tag{62}$$

as a function of the angle θ between the two polarization vectors reveals that the component S_{x_0} is squeezed over a wide range of θ as is evident from the figure (2) and figure (3) shown below. The graphical study also reveals that squeezing appears only when the degree of polarization of both the spins are more than 77% of that for a pure state in each case. In particular, when the states are pure, the combined system will also be in a pure state but the spin 1 projection of this pure state will be in an entangled state (refer to equation (25) in reference [10]). In this state, the squeezing condition reduces to

$$\cos^2 2\theta < |\cos 2\theta| \tag{63}$$

which agrees with the result obtained in our earlier paper [10] (except that we have called the angle between $\vec{P}(1)$ and $\vec{P}(2)$ as 2θ here, while it is taken as θ there). The origin of the squeezing behaviour of the coupled spin 1 system can be traced as arising due to the intrinsic quantum correlations that exist between the individual spinors. These correlations can be classified as (1) those that arise due to the coupling of the two subsystems and (2) those that arise when the combined total density matrix ρ_C is projected on to the desired spin space. In our present case, we have taken ρ_C to be a direct product of the two subsystem density matrices $\rho(1)$ and $\rho(2)$. Such a ρ_C is not entangled and therefore there are no correlations of the first kind. However, when we take the spin 1 projection of ρ_C, the correlations of the second type will appear in the spin 1 projection. These correlations are given by

$$C_{xx}^{12} = \frac{P_s^2 - P_d(P(1)^2+P(2)^2) - 2P(1)^2P(2)^2(1+\sin^2\theta\cos 2\phi)}{4(3+P_d)P_s^2} \tag{64}$$

$$C_{yy}^{12} = \frac{P_s^2 - 2P(1)^2P(2)^2(1-\sin^2\theta\cos 2\phi) - P_d(P(1)^2+P(2)^2)}{4(3+P_d)P_s^2} \tag{65}$$

$$C_{xz}^{12} = \frac{|\vec{P}(1)\times\vec{P}(2)|\,(P(2)^2-P(1)^2)\cos\phi}{2(3+P_d)P_s^2} \tag{66}$$

$$C_{zz}^{12} = \frac{1}{12} - \frac{P_s^2}{(3+P_d)^2} + \frac{P_n}{3(3+P_d)P_s^2} \tag{67}$$

$$C_{zy}^{12} = \frac{(P(1)^2-P(2)^2)|\vec{P}(1)\times\vec{P}(2)|\sin\phi}{2(3+P_d)P_s^2} \tag{68}$$

$$C_{xy}^{12} = 0\,, \tag{69}$$

Figure 4. Variation of spin-spin correlations C_{xx} $(+)$, C_{yy} $(\bullet)$, C_{zz} $(\times)$, C_{xz} $(\circ)$, C_{yz} $(\star)$ and squeezing Q $(\diamond)$ with respect to θ, for $P(1) = 0.9,\ P(2) = 0.85$ and $\phi = 0°$.

Figure 5. Variation of spin-spin correlations C_{xx} $(+)$, C_{yy} $(\bullet)$, C_{zz} $(\times)$, C_{xz} $(\circ)$, C_{yz} $(\star)$ and squeezing Q $(\diamond)$ with respect to θ, for $P(1) = 0.95,\ P(2) = 0.92$ and $\phi = 5°$.

Figure 6. Variation of spin-spin correlations C_{xx} $(+)$, C_{yy} $(\bullet)$, C_{zz} $(\times)$, C_{xz} $(\circ)$, C_{yz} $(\star)$ and squeezing Q $(\diamond)$ with respect to θ, for $P(1) = 0.85,\ P(2) = 0.95$ and $\phi = 10°$.

where $P_s = |\vec{P}(1)+\vec{P}(2)|$, $P_n = 4P(1)^2P(2)^2+2\vec{P}(1)\cdot\vec{P}(2)(P(1)^2+P(2)^2)-\sin^2\theta$ and $P_d = \vec{P}(1)\cdot\vec{P}(2)$. We have done a detailed graphical study of the correlations and squeezing for various values of the independent parameters. While the study reveals that squeezing and correlations coexist and are equally more pronounced in certain ranges, there are also narrow regions where one exists in the absence of the other. All these aspects are revealed in the figures 4-6.

It is therefore of interest to study more general cases of coupling of the sub systems in order to identify definite relationship between correlations and squeezing. In the context of quantum computation, the nature of coupled states has been studied [22] under the following configurations:

(1) ρ_C is strongly separable ; i.e., $\rho_C = \rho(1) \otimes \rho(2)$
(2) ρ_C is weakly separable ; i.e., $\rho_C = \sum p_i\ \rho_i(1) \otimes \rho_i(2)$, $\sum p_i = 1$, $p_i \geq 0$
(3) ρ_C is non-separable ; i.e., ρ_C cannot be expressed as in (1) and (2).

The third configuration is indeed recognized as possessing quantum entanglement. We have discussed the strongly separable mixed state case in this paper for the particular case of $s_1 = \frac{1}{2}$ and $s_2 = \frac{1}{2}$. We wish to look at the squeezing and the correlation aspects for the cases (2) and (3) in a sequel to this paper.

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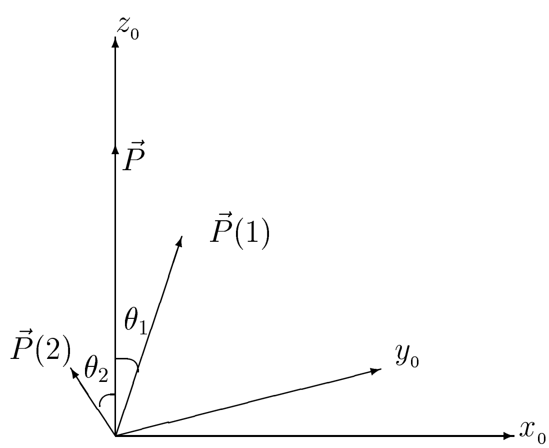
Table 1. Squeezed spin states specified by their non-zero spherical tensor parameters in LF and variances in S_{x_0} and S_{y_0}.

Spin value	t^2_0	t^2_2	t^1_0	$\Delta S^2_{x_0}$	$\Delta S^2_{y_0}$	$\frac{1}{2}\lvert\langle S_{z_0}\rangle\rvert$
$3/2$	0.9	0.3	1.25	1.17	0.34	0.7
$3/2$	0.7	0.5	1.06	1.5	0.28	0.6
$3/2$	0.61	0.49	0.99	1.54	0.34	0.55
$3/2$	0.41	0.63	0.81	1.82	0.27	0.45
1	0.7	0.65	0.8	0.876	0.12	0.12
1	0.5	0.45	0.9	0.81	0.28	0.37
1	0.4	0.65	0.5	0.94	0.197	0.204
1	0.3	0.49	0.7	0.83	0.27	0.286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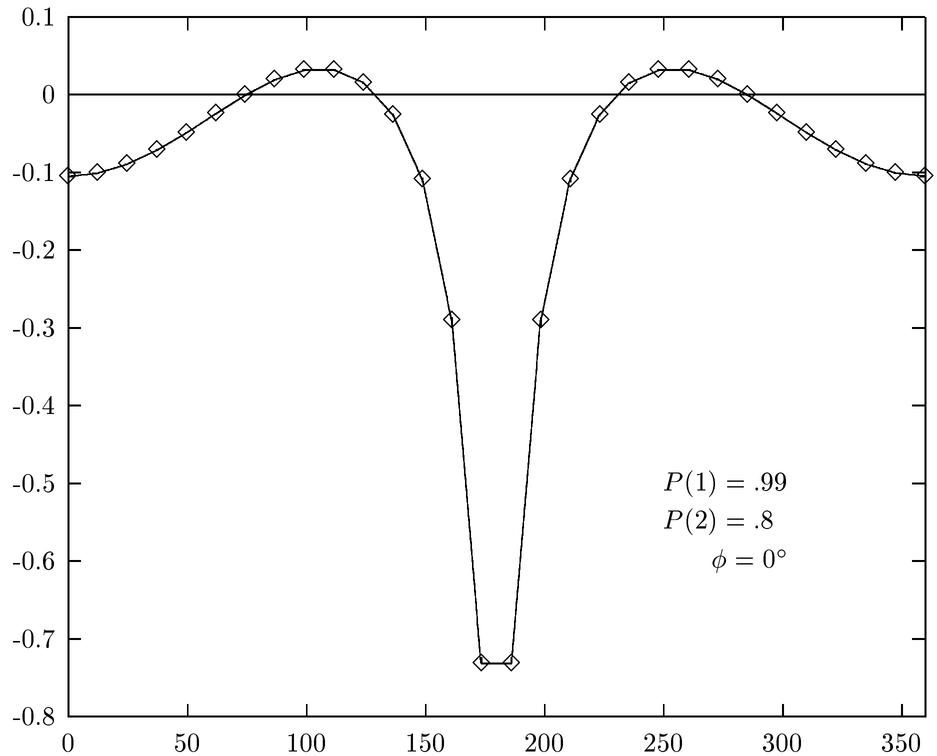
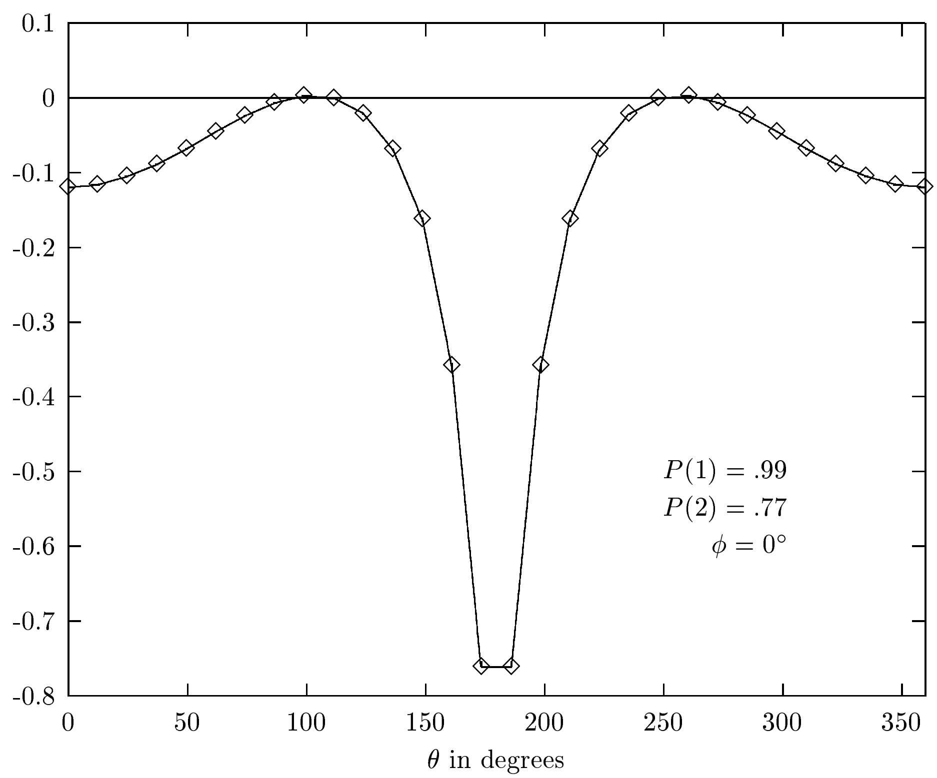
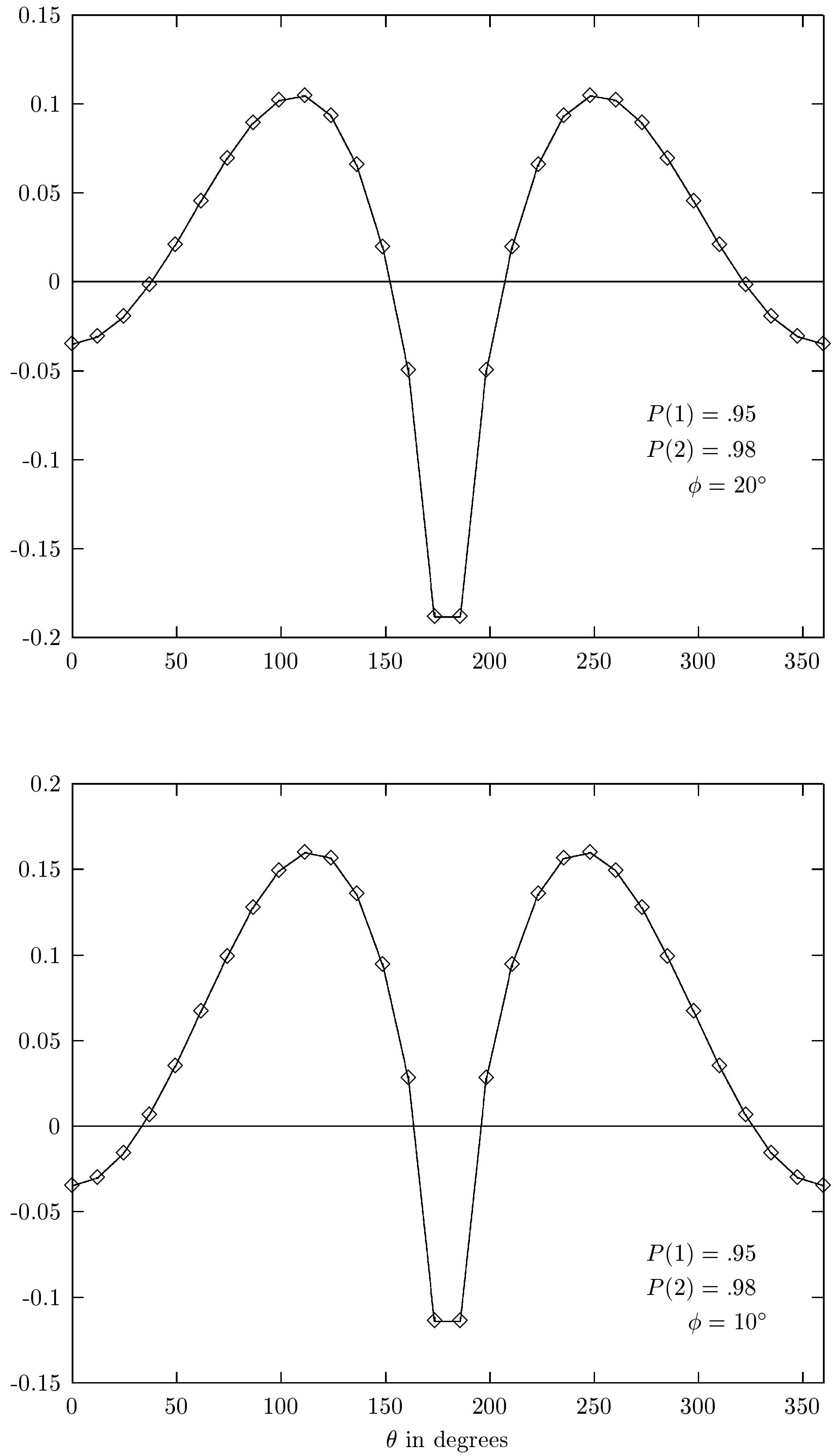
$\vec{P}$
$\vec{P}(1)$
θ_1
$\vec{P}(2)$
θ_2
y_0
x_0



$P(1) = .99$
$P(2) = .8$
$\phi = 0°$



$P(1) = .99$
$P(2) = .77$
$\phi = 0°$
θ in degrees



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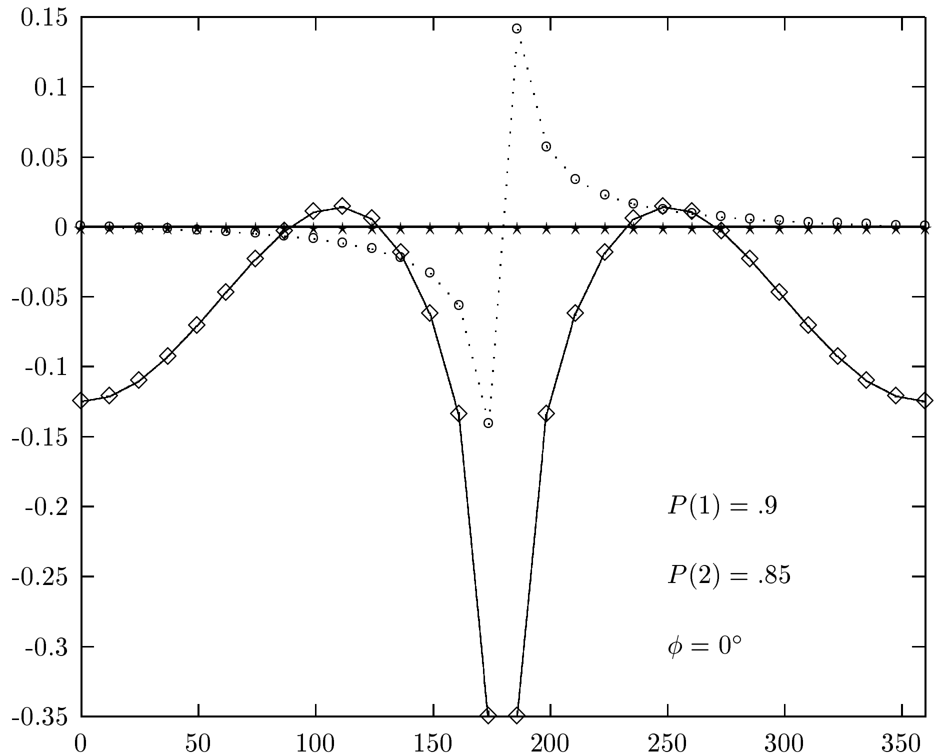
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
0
50
100
150
200
250
300
350
$P(1) = .95$
$P(2) = .98$
$\phi = 20°$
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
0
50
100
150
200
250
300
350
$P(1) = .95$
$P(2) = .98$
$\phi = 10°$
θ in degrees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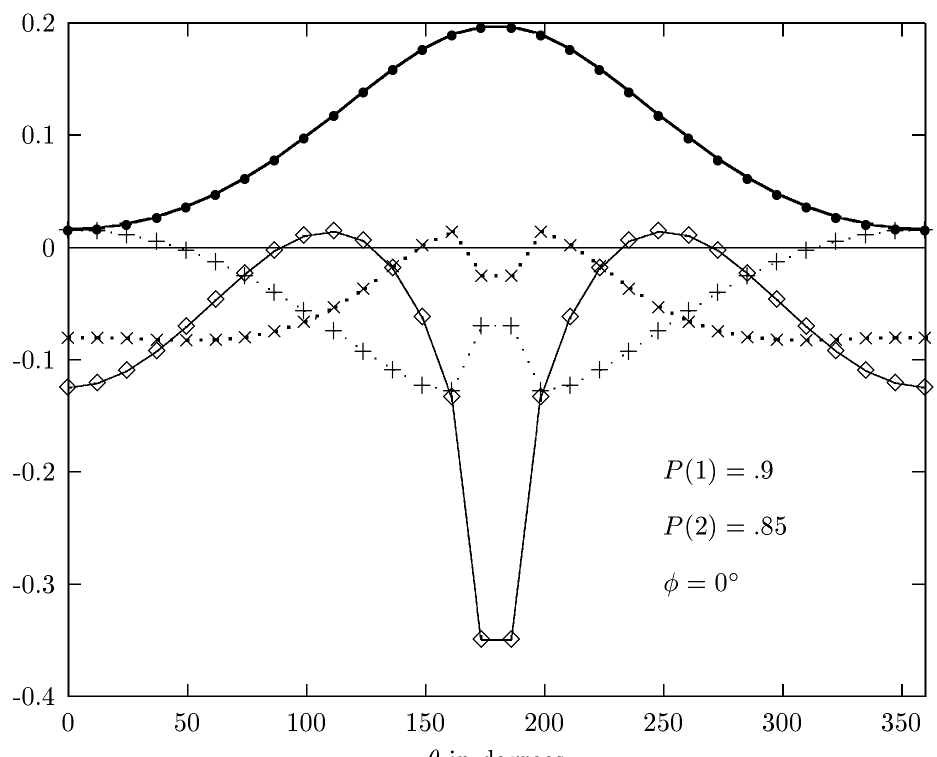
0
50
100
150
200
250
300
350
$P(1) = .9$
$P(2) = .85$
$\phi = 0°$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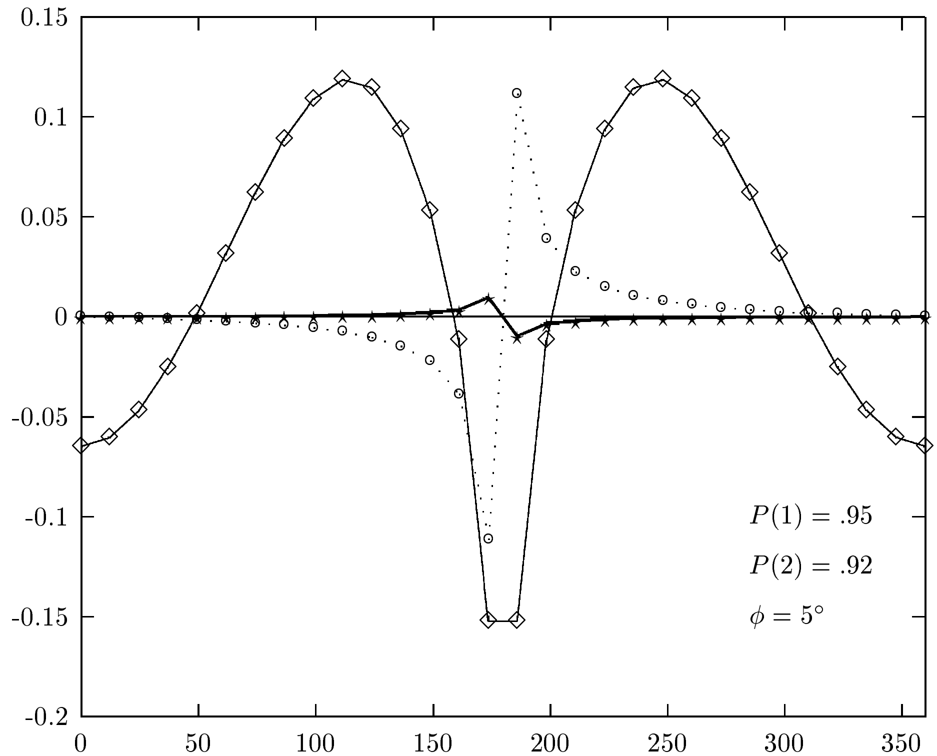
$P(1) = .9$
$P(2) = .85$
$\phi = 0°$



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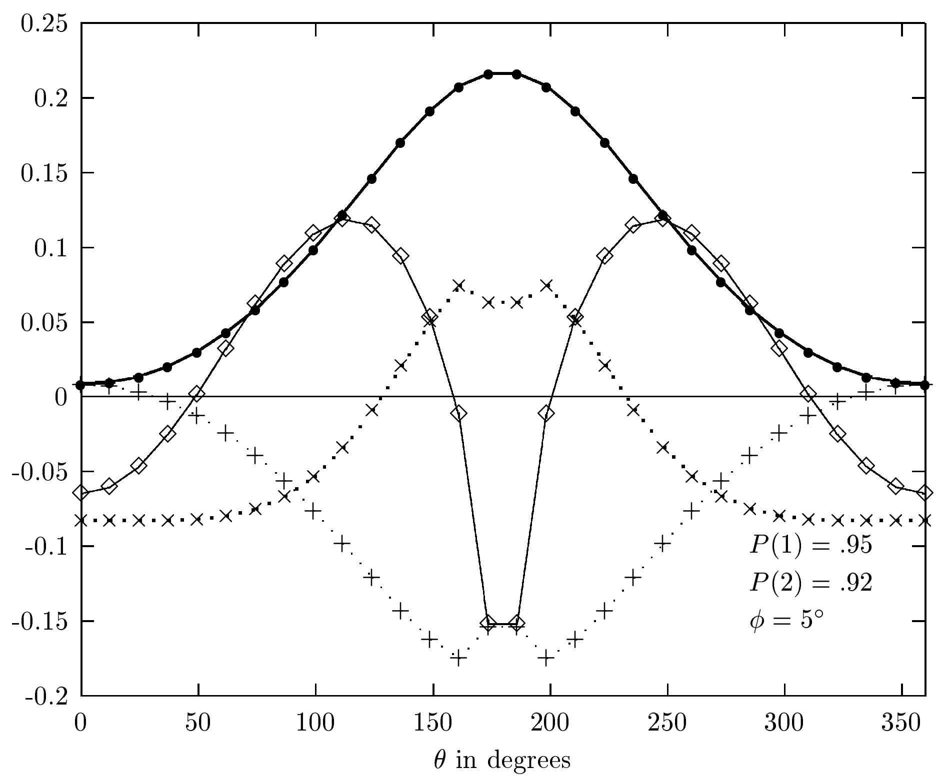
$P(1) = .95$
$P(2) = .92$
$\phi = 5°$



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θ in degrees
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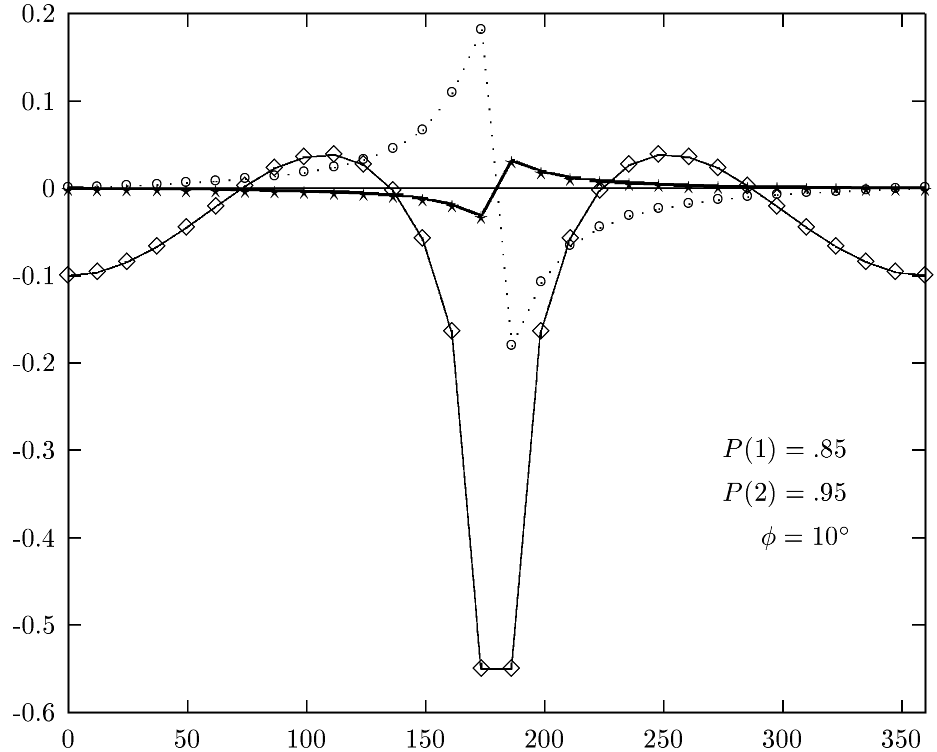
$P(2) = .92$
$\phi = 5°$



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$P(1) = .85$
$P(2) = .95$
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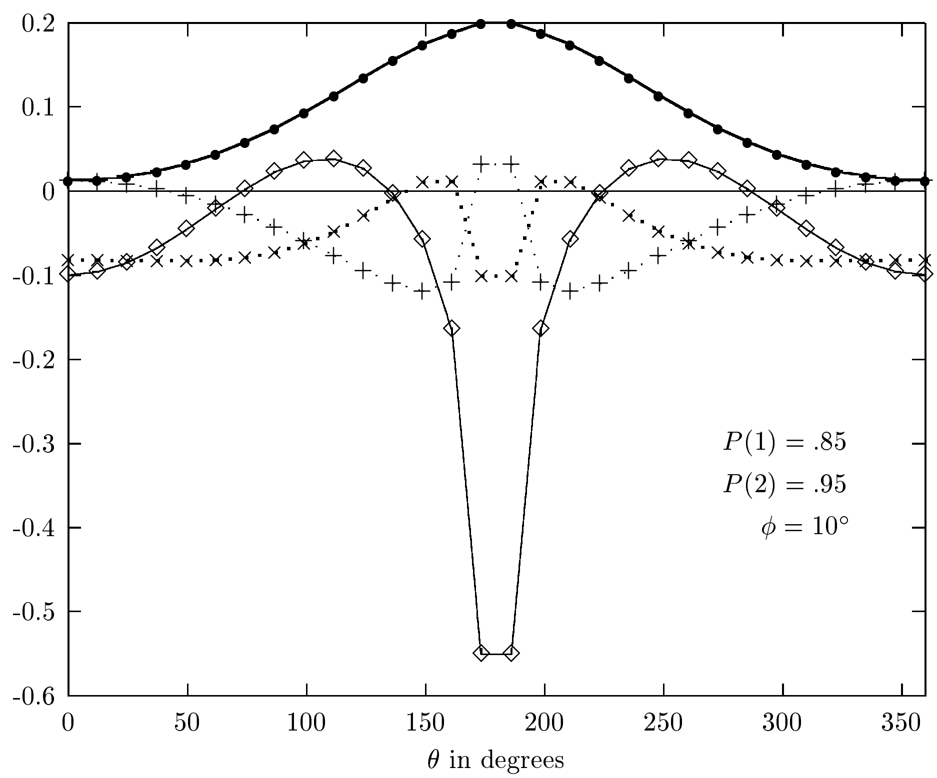

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150
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250
300
350
θ in degrees
$P(1) = .85$
$P(2) = .95$
$\phi = 10°$